“Path-dependent Option Valuation under Jump-diffusion Processes”

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Topics

Why jump-diffusion models?

Review of the vanilla Euro-style Valuation

A “Standard Machine” for Path-dependent Options:

a. Perpetual American
b. Knock-outs (Down-and-out call, etc.)
c. One-touch Options (Rebate term)
Why Jump-Diffusion Models for Options?

I. Benchmark model (exponential Brownian motion):

Attractive features:
• limited liability stock prices
• uncorrelated, level independent returns
• simple formulas (methods) for option prices (euro, amer)

Weak points:
• Actual stock price distributions have wider tails
• Lacks volatility clustering (auto-corr. of absolute returns)
• Lacks stock price jumps
• Poor fit to real-world option prices (smile/skew)

II. Jump-diffusion processes (exponential Lévy processes)
- Stationary, independent increment processes
- Continuous-time analog of Random Walk
- Brownian motion plus Poisson-driven jump process

Attractive features:
• all the attractive benchmark features +
• large flexible class of models, each with a few parameters
• wide return tails common (exponential decay, moments)
• Good fits to expiring options (fear of jumps/crashes?)

Weak points:
• Lacks volatility clustering (auto-corr. of absolute returns)
• Brownian motion $\approx$ Large number of small jumps
Stock price Evolution and Examples

\[ S_t = S_0 \exp(X_t), \]

(Assumption: this is under the martingale pricing measure \( Q \))

where \( X_t = ct + \sigma B_t + \Delta X_t; \quad \Delta X_t = \sum_{i=1}^{n(t)} y_i \)

Jump probability(t) \( \approx \lambda \Delta t \quad y \sim p(y) \) (Jump distribution)

Examples of Jump distributions:

(A.1) Merton’s 1976 jump-diffusion model with log-normally distributed jumps:

\[ p(y) = \frac{1}{\sqrt{2\pi\delta^2}} \exp \left[ -\frac{(y - \mu_J)^2}{2\delta^2} \right] \]

(A.2) Degenerate Case: Point-jump:

\[ p(y) = \delta(y - \mu_J) \]

Typical SPX Smile Fit:
\( \lambda \approx 0.3, \quad \mu_J \approx -0.25, \quad \delta \approx 0.10 \)
Figure 1

SPX Options: Implied Volatility vs. Strike on Aug. 16, 2002 (1 month to Expiration)

Constant Volatility + Jumps

SPX ↓
Vanilla European-style options
Solutions in “Fourier” space are simple

Ingredients:

1. The generalized Fourier transform of the payoff function:
   For the call option:
   \[
   \hat{g}(z) = \int_{-\infty}^{\infty} e^{-ix} (e^x - K)^+ \, dx = -\frac{K^{1-iz}}{(z^2 + iz)}, \quad \text{Im} \, z < 1
   \]

2. The characteristic function of the Lévy process:
   \[
   \varphi_T(z) = \int_{-\infty}^{\infty} e^{izx} \mu_T(x) \, dx = \mathbb{E}[\exp(izX_T)] = \exp(-T\Psi(z))
   \]
   where \( \Psi(z) \) is the “characteristic exponent”.
   For the Point Jump model:
   \[
   \psi(z) = -iz\omega + \frac{1}{2}z^2\sigma^2 - \lambda\{\exp(i\mu_Jz) - 1\} \quad \text{(Entire)}
   \]

3. Finally, the Call Option price is given by:
   \[
   C(S_0, K, T) = -\frac{e^{-rT}}{2\pi} K \int_{\text{Im} \, z < -1} \left( \frac{S_0}{K} \right)^{iz} e^{-T\psi(z)} \frac{1}{z(z + i)} \, dz,
   \]
   The integration is along a line parallel to the real z-axis.
European-style options (cont.)

The solution is very easy to derive and “obvious”:

First, we need the inversion formula for the payoff function:

\[ g(x) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{izx} \hat{g}(z) dz, \quad x = \log S_T, \quad z \in \text{Payoff strip} \]

Then, by martingale pricing:

\[
C(S_0) = e^{-rT} \mathbb{E} \left[ g(\log S_T) \right] = \frac{e^{-r\tau}}{2\pi} \mathbb{E} \left[ \int_{i\nu-\infty}^{i\nu+\infty} (S_T)^{iz} \hat{g}(z) dz \right]
\]

\[
= \frac{e^{-r\tau}}{2\pi} \mathbb{E} \left[ \int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz} e^{izX_T} \hat{g}(z) dz \right]
\]

\[
= \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz} e^{-T\psi(z)} \hat{g}(z) dz ,
\]

Ok to exchange the integrations (sufficient conditions) if:

1. \( g(x) \) is Fourier integrable in some Payoff strip \( S_g \) and bounded for \( |x| < \infty \).
2. \( \psi(z) \) is regular in some strip \( S_X : \alpha < \text{Im} z < \beta \)
3. \( \nu = \text{Im} z \) lies in the intersection of these two strips
A Standard Machine:
the Down-and-out Call (or Down-and-out “anything”)

How we will do it.

1. Write the payoff in terms of its Fourier Transform
2. Write the barrier condition using a representation for \(1_{\{Y > 0\}}\).
4. Find a “Fluctuation Identity” to do the expectation.
5. Done with General Formula!

This general procedure works for all the problems I listed at the beginning and probably lots of others.

It saves having to learn a lot of the “heavy machinery” of The Boyarchenko/Levendorskii approach.

6. Then, for your particular model:
   Try to do as many integrals as possible analytically;
   (Residue Calculus).

7. Do the remaining integrals numerically.
1. **Write the payoff in terms of its Fourier Transform**

Minimum Process: \[ S_T = \min_{0 \leq t \leq T} S_t \]

\[ C_{DOC}(S_0, K, H, T) = e^{-rT} E \left[ (S_T - K)^+ 1_{S_T > H} \right] \]

or, with \( x = \log S_0, \ h = \log H, \) and \( N_T = \min_{0 \leq t \leq T} X_t \)

\[ f_{DOC}(x, T) = e^{-rT} E \left[ (e^{x + X_T} - K)^+ 1_{N_T > h - x} \right] \]

\[ = e^{-rT} E \left[ \int_{\text{Im } z < -1} \frac{dz}{2\pi} \exp \left\{ iz(x + X_T) \right\} \hat{g}(z) 1_{N_T > h - x} \right] \]
2. Write the barrier condition using a representation for $1_{\{Y>0\}}$.

$$1_{\{Y>0\}} = \int_{\text{Im} \xi < 0} \frac{d \xi}{2\pi i \xi} \exp(i \xi Y),$$

where $Y = x - h + N_T$.

This produces:

$$f_{DO}(x,T) = e^{-rT} \times$$

$$E \left[ \int_{\text{Im} z < -1} \frac{dz}{2\pi} \int_{\text{Im} \xi < 0} \frac{d \xi}{2\pi i \xi} \hat{g}(z) \exp(i z x + i \xi (x - h)) \exp(i x T + i \xi N_T) \right]$$


We need a formula for $E[\exp(i z X_T + i \xi N_T)]$. 
4. Find a “Fluctuation Identity” to do the expectation

Some Fluctuation Identities:

1. Factorization identities: (Spitzer, Rogozin, others)

\[
\frac{q}{q + \psi(z)} = \phi_q^+(z) \phi_q^-(z),
\]

where

\[
\phi_q^+(z) = E[\exp(i z M_{\tau(q)})] = q \int_0^\infty e^{-qt} E[e^{iz M_t}] \, dt
\]

\[
\phi_q^-(z) = E[\exp(i z N_{\tau(q)})] = q \int_0^\infty e^{-qt} E[e^{iz N_t}] \, dt
\]

\(\tau(q)\) is an independent (exponentially distributed) random stopping time.

Not computationally effective. But, this one is:

\[
\phi_q^-(\xi) = \exp\left\{ \int_{(\text{Im } \xi)^+}^{\sigma^+} \frac{d\eta}{-2\pi i} \frac{\xi \log[q + \psi(\eta)]}{\eta (\xi - \eta)} \right\}
\]
4. Find a “Fluctuation Identity” to do the expectation

Some Fluctuation Identities:

2. Here’s the one we really need:

\[
q \int_0^{\infty} e^{-qT} E[\exp(i z X_T + i \xi N_T)] dT = \phi_q^+(z) \phi_q^-(\xi + z)
\]

By Laplace Inversion:

\[
E[\exp(i z X_T + i \xi N_T)] = \int_{\text{Re} q > r} \frac{dq}{2\pi i q} e^{qT} \phi_q^+(z) \phi_q^-(\xi + z)
\]
5. Done with General Formula!

Here it is:

\[ f_{DO}(x,T) = \int_{\text{Re}q > \text{r}} \frac{dq}{2\pi i} e^{qT} F_{DO}(x,q). \]

where for the call option payoff, it reads

\[ F_{DOC}(x,q) = \left( \frac{K e^{-rT}}{q} \right) \times \]

\[ \int_{C_1} \frac{d\xi}{2\pi i} \int_{C_2} \frac{dz}{2\pi i} \exp \left\{ iz(h-k) + i\xi(x-h) \right\} \frac{\phi_q^+(z)\phi_q^-(\xi)}{(z-\xi)z(z+i)} \]

Integration contours:

\[ C_1 : \text{Im} \lambda^- < \text{Im} \xi < -1, \]
\[ C_2 : \text{Im} \alpha_1(q) < \text{Im} z < -1 \text{ and } \text{Im} \xi < \text{Im} z \]
6. Then, for your particular model:
   Try to do as many integrals as possible analytically;
   (Residue Calculus).

7. Do the remaining integrals numerically.

**Example 1:** Suppose your model has “No negative jumps”.
(This means the Barrier is crossed continuously).

When there are no negative jumps, the Laplace transform of the
down-and-out call option is given by: \((x > h)\)

\[
F_{DOC}(x, q)\big|_{NNJ} = F_E(x, q) - \exp \{i\gamma(q)(x - h)\} F_E(h, q),
\]

where \(F_E(x, q)\) is the Laplace transform of the European
(no barrier) call option:

\[
F_E(x, q) = \int_{\text{Im} \alpha_1(q) < \text{Im} z < -1} \frac{dz}{2\pi} \frac{-Ke^{-rT}}{z(z + i)(q + \psi(z))} \exp \{iz(x - k)\}.
\]

(This one could be proved directly from the PIDE).
6. Then, for your particular model:
   Try to do as many integrals as possible analytically;
   (Residue Calculus).

7. Do the remaining integrals numerically.

**Example 2:** Suppose your model has a “Negative Point Jump”.
Then, you have to investigate the roots (zeros) of

$$q + \psi(z) = 0,$$

where

$$q + \psi(z) = q - iz\omega + \frac{1}{2} z^2 \sigma^2 - \lambda(\exp(\mu, jz) - 1)$$

It turns out there are (almost certainly) an infinity of these in the complex $z$-plane. Then, I have a (conjectured) result using these roots:

$$F_{DOC}(x, q) = \frac{ie^{x-rT}}{\alpha(\alpha + i)} \left( \frac{1}{\psi'(\alpha)} + \lim_{N_\gamma \to \infty} \sum_{i=1}^{N_\gamma} \frac{\exp\{i(\gamma_i - \alpha)(x - h)\}}{\psi'(\gamma_i)} \right)$$

$\alpha$ is the single lower half-plane root. (Im $\alpha < 0$)
$\gamma_i$ are the infinity of upper half-plane roots. (Im $\gamma_i > 0$)
1. A plot of $|f'(z)/f(z)|$ where $f(z) = r + \psi(z)$ for the Merton jump-diffusion
Figure 2.
The location of some roots of $r + \psi(z) = 0$ for the point jump model.