

**USC Math. Finance**  
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**“Path-dependent Option Valuation  
under Jump-diffusion Processes”**

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posted at [www.optioncity.net](http://www.optioncity.net) (Publications)**

**Topics**

**Why jump-diffusion models?**

**Review of the vanilla Euro-style Valuation**

**A “Standard Machine” for Path-dependent Options:**

- a. Perpetual American**
- b. Knock-outs (Down-and-out call, etc.)**
- c. One-touch Options (Rebate term)**

# Why Jump-Diffusion Models for Options?

## I. Benchmark model (exponential Brownian motion):

### Attractive features:

- limited liability stock prices
- uncorrelated, level independent returns
- simple formulas (methods) for option prices (euro,amer)

### Weak points:

- Actual stock price distributions have wider tails
- Lacks volatility clustering (auto-corr. of absolute returns)
- Lacks stock price jumps
- Poor fit to real-world option prices (smile/skew)

## II. Jump-diffusion processes (exponential Lévy processes)

- Stationary, independent increment processes
- Continuous-time analog of Random Walk
- Brownian motion plus Poisson-driven jump process

### Attractive features:

- all the attractive benchmark features +
- large flexible class of models, each with a few parameters
- wide return tails common (exponential decay, moments)
- Good fits to expiring options (fear of jumps/crashes?)

### Weak points:

- Lacks volatility clustering (auto-corr. of absolute returns)
- Brownian motion  $\approx$  Large number of small jumps

## Stock price Evolution and Examples

$$S_t = S_0 \exp(X_t),$$

(Assumption: this is under the martingale pricing measure  $Q$ )

$$\text{where } X_t = ct + \sigma B_t + \Delta X_t; \quad \Delta X_t = \sum_{i=1}^{n(t)} y_i$$

Jump probability(t)  $\approx \lambda \Delta t$   $y \sim p(y)$  (Jump distribution)

**Examples of Jump distributions:**

**(A.1) Merton's 1976 jump-diffusion model with log-normally distributed jumps:**

$$p(y) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-(y - \mu_J)^2 / 2\delta^2\right]$$

**(A.2) Degenerate Case: Point-jump:**

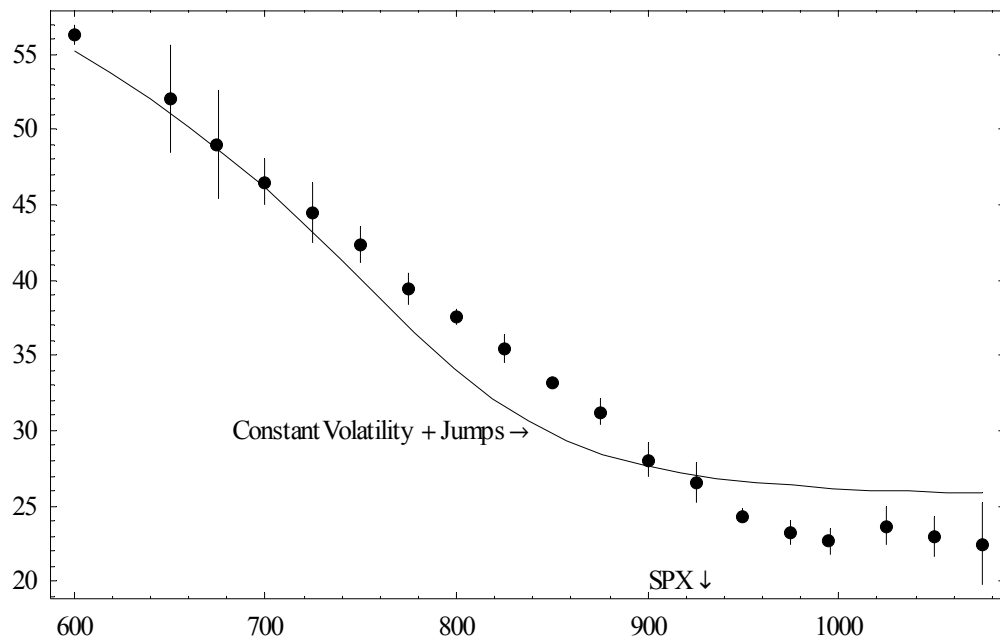
$$p(y) = \delta(y - \mu_J)$$

**Typical SPX Smile Fit:**

$$\lambda \approx 0.3, \quad \mu_J \approx -0.25, \quad \delta \approx 0.10$$

**Figure 1**

SPX Options: Implied Volatility vs. Strike on Aug. 16, 2002 (1 month to Expiration)



## Vanilla European-style options Solutions in “Fourier” space are simple

### Ingredients:

1. The generalized Fourier transform of the payoff function:  
For the call option:

$$\hat{g}(z) = \int_{-\infty}^{\infty} e^{-izx} (e^x - K)^+ dx = -\frac{K^{1-iz}}{(z^2 + iz)}, \quad \text{Im } z < 1$$

2. The characteristic function of the Lévy process:

$$\varphi_T(z) = \int_{-\infty}^{\infty} e^{izx} p_T(x) dx = \mathbb{E}[\exp(izX_T)] = \exp(-T\Psi(z))$$

where  $\Psi(z)$  is the “characteristic exponent”.

For the Point Jump model:

$$\psi(z) = -iz\omega + \frac{1}{2}z^2\sigma^2 - \lambda\{\exp(i\mu_J z) - 1\} \quad (\text{Entire})$$

3. Finally, the Call Option price is given by :

$$C(S_0, K, T) = -\frac{e^{-rT}}{2\pi} K \int_{\text{Im } z < -1} \left(\frac{S_0}{K}\right)^{iz} \frac{e^{-T\psi(z)}}{z(z+i)} dz,$$

The integration is along a line parallel to the real z-axis.

## European-style options (cont.)

The solution is very easy to derive and “obvious”:

First, we need the inversion formula for the payoff function:

$$g(x) = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{izx} \hat{g}(z) dz, \quad x = \log S_T, \quad z \in \text{Payoff strip}$$

Then, by martingale pricing:

$$\begin{aligned} C(S_0) &= e^{-r\tau} \mathbb{E}[g(\log S_T)] = \frac{e^{-r\tau}}{2\pi} \mathbb{E} \left[ \int_{i\nu-\infty}^{i\nu+\infty} (S_T)^{iz} \hat{g}(z) dz \right] \\ &= \frac{e^{-r\tau}}{2\pi} \mathbb{E} \left[ \int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz} e^{izX_T} \hat{g}(z) dz \right] \\ &= \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz} e^{-T\psi(z)} \hat{g}(z) dz, \end{aligned}$$

Ok to exchange the integrations (sufficient conditions) if:

1.  $g(x)$  is Fourier integrable in some Payoff strip  $S_g$  and bounded for  $|x| < \infty$ .
2.  $\psi(z)$  is regular in some strip  $S_X: \alpha < \text{Im } z < \beta$
3.  $\nu = \text{Im } z$  lies in the intersection of these two strips

**A Standard Machine:  
the Down-and-out Call (or Down-and-out “anything”)**

**How we will do it.**

- 1. Write the payoff in terms of its Fourier Transform**
- 2. Write the barrier condition using a representation for  $1_{\{Y>0\}}$ .**
- 3. Bring an expectation inside an integral.**
- 4. Find a “Fluctuation Identity” to do the expectation.**
- 5. Done with General Formula!**

**This general procedure works for all the problems I listed at the beginning and probably lots of others.**

**It saves having to learn a lot of the “heavy machinery” of The Boyarchenko/Levendorskii approach.**

- 6. Then, for your particular model:  
    Try to do as many integrals as possible analytically;  
    (Residue Calculus).**
- 7. Do the remaining integrals numerically.**

## Down-and-out Call (or Down-and-out “anything”) (cont).

### 1. Write the payoff in terms of its Fourier Transform

**Minimum Process:**  $\underline{S}_T = \min_{0 \leq t \leq T} S_t$

$$C_{DOC}(S_0, K, H, T) = e^{-rT} E \left[ (S_T - K)^+ \mathbf{1}_{\underline{S}_T > H} \right]$$

or, with  $x = \log S_0$ ,  $h = \log H$ , and  $N_T = \min_{0 \leq t \leq T} X_t$

$$f_{DOC}(x, T) = e^{-rT} E \left[ (e^{x+X_T} - K)^+ \mathbf{1}_{N_T > h-x} \right]$$

$$= e^{-rT} E \left[ \int_{\text{Im } z < -1} \frac{dz}{2\pi} \exp \{ iz(x + X_T) \} \hat{g}(z) \mathbf{1}_{N_T > h-x} \right]$$

## Down-and-out Call (or Down-and-out “anything”) (cont)

**2. Write the barrier condition using a representation for  $1_{\{Y>0\}}$ .**

$$1_{Y>0} = \int_{\text{Im}\xi < 0} \frac{d\xi}{2\pi i \xi} \exp(i\xi Y),$$

where  $Y = x - h + N_T$ .

**This produces:**

$$f_{DO}(x, T) = e^{-rT} \times$$

$$E \left[ \int_{\text{Im}z < -1} \frac{dz}{2\pi} \int_{\text{Im}\xi < 0} \frac{d\xi}{2\pi i \xi} \hat{g}(z) \exp(izx + i\xi(x - h)) \exp(izX_T + i\xi N_T) \right]$$

**3. Bring an expectation inside an integral.**

We need a formula for  $E[\exp(izX_T + i\xi N_T)]$ .

#### 4. Find a “Fluctuation Identity” to do the expectation

**Some Fluctuation Identities:**

##### 1. Factorization identities: (Spitzer, Rogozin, others)

$$\frac{q}{q + \psi(z)} = \phi_q^+(z)\phi_q^-(z),$$

where

$$\phi_q^+(z) = E[\exp(izM_{\tau(q)})] = q \int_0^\infty e^{-qt} E[e^{izM_t}] dt$$

$$\phi_q^-(z) = E[\exp(izN_{\tau(q)})] = q \int_0^\infty e^{-qt} E[e^{izN_t}] dt$$

$\tau(q)$  is an independent (exponentially distributed) random stopping time.

Not computationally effective. But, this one is:

$$\phi_q^-(\xi) = \exp\left\{ \int_{(\text{Im } \xi)^+ < \text{Im } \eta < \sigma^+} \frac{d\eta}{(-2\pi i)} \frac{\xi \log[q + \psi(\eta)]}{\eta(\xi - \eta)} \right\}$$

#### **4. Find a “Fluctuation Identity” to do the expectation**

**Some Fluctuation Identities:**

**2. Here’s the one we really need:**

$$q \int_0^{\infty} e^{-qT} E[\exp(izX_T + i\xi N_T)] dT = \phi_q^+(z) \phi_q^-(\xi + z)$$

**By Laplace Inversion:**

$$E[\exp(izX_T + i\xi N_T)] = \int_{\text{Re } q > r} \frac{dq}{2\pi i q} e^{qT} \phi_q^+(z) \phi_q^-(\xi + z)$$

## 5. Done with General Formula!

Here it is:

$$f_{DO}(x, T) = \int_{\text{Re } q > r} \frac{dq}{2\pi i} e^{qT} F_{DO}(x, q).$$

where for the call option payoff, it reads

$$F_{DOC}(x, q) = \left( \frac{Ke^{-rT}}{q} \right) \times \int_{C_1} \frac{d\xi}{2\pi i} \int_{C_2} \frac{dz}{2\pi} \frac{\exp\{iz(h-k) + i\xi(x-h)\}}{(z-\xi)z(z+i)} \phi_q^+(z) \phi_q^-(\xi)$$

Integration contours:

$$C_1 : \text{Im } \lambda^- < \text{Im } \xi < -1,$$

$$C_2 : \text{Im } \alpha_1(q) < \text{Im } z < -1 \text{ and } \text{Im } \xi < \text{Im } z$$

**6. Then, for your particular model:  
Try to do as many integrals as possible analytically;  
(Residue Calculus).**

**7. Do the remaining integrals numerically.**

**Example 1: Suppose your model has “No negative jumps”.  
(This means the Barrier is crossed continuously).**

**When there are no negative jumps, the Laplace transform of the  
down-and-out call option is given by: ( $x > h$ )**

$$F_{DOC}(x, q)|_{NNJ} = F_E(x, q) - \exp\{i\gamma(q)(x - h)\} F_E(h, q),$$

**where  $F_E(x, q)$  is the Laplace transform of the European  
(no barrier) call option:**

$$F_E(x, q) = \int_{\text{Im } \alpha_1(q) < \text{Im } z < -1} \frac{dz}{2\pi} \frac{-Ke^{-rT}}{z(z+i)(q+\psi(z))} \exp\{iz(x-k)\}.$$

**(This one could be proved directly from the PIDE).**

**6. Then, for your particular model:**

**Try to do as many integrals as possible analytically;  
(Residue Calculus).**

**7. Do the remaining integrals numerically.**

**Example 2: Suppose your model has a “Negative Point Jump”.**

**Then, you have to investigate the roots (zeros) of**

$$q + \psi(z) = 0,$$

**where**

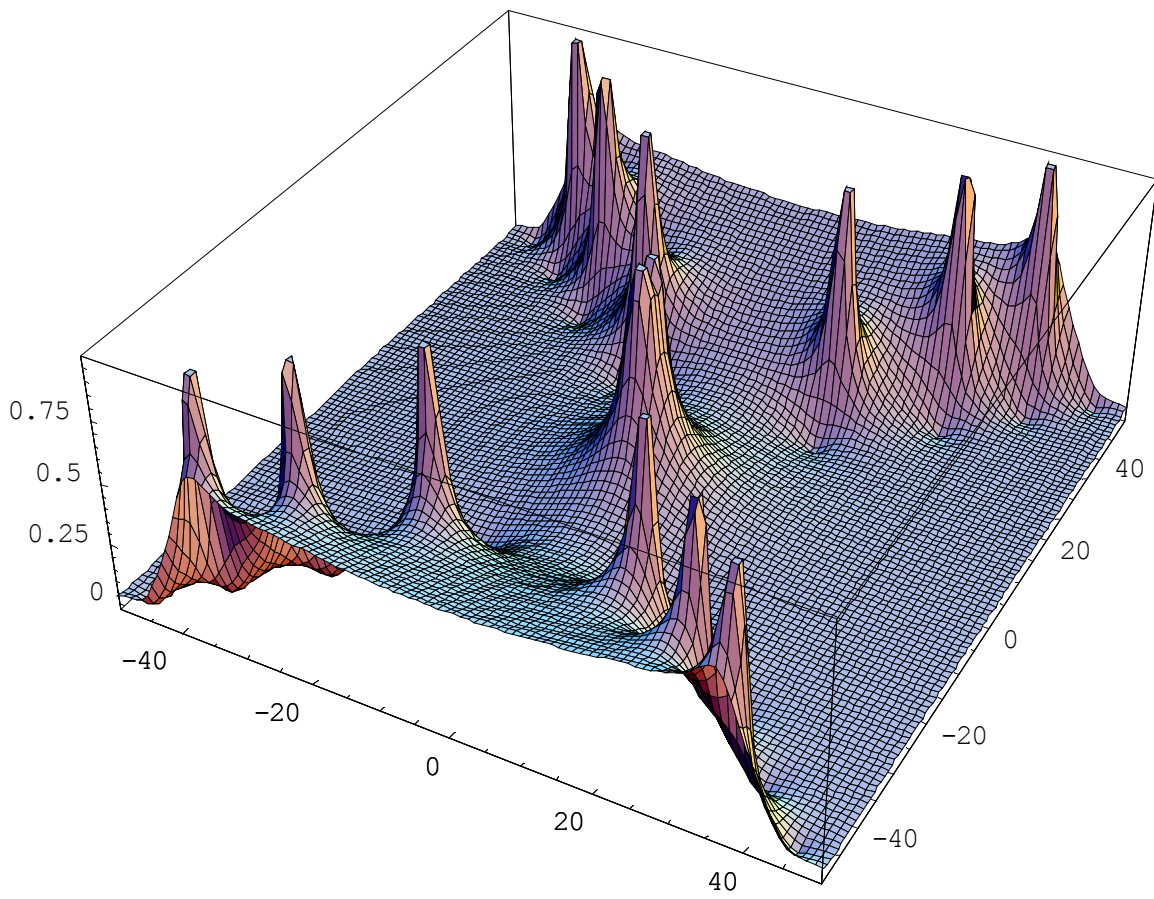
$$q + \psi(z) = q - iz\omega + \frac{1}{2}z^2\sigma^2 - \lambda(\exp(i\mu_J z) - 1)$$

**It turns out there are (almost certainly) an infinity of these in the complex z-plane. Then, I have a (conjectured) result using these roots:**

$$F_{DOC}(x, q) = \frac{i e^{x-rT}}{\alpha(\alpha + i)} \left( \frac{1}{\psi'(\alpha)} + \lim_{N_\gamma \rightarrow \infty} \sum_{i=1}^{N_\gamma} \frac{\exp\{i(\gamma_i - \alpha)(x - h)\}}{\psi'(\gamma_i)} \right)$$

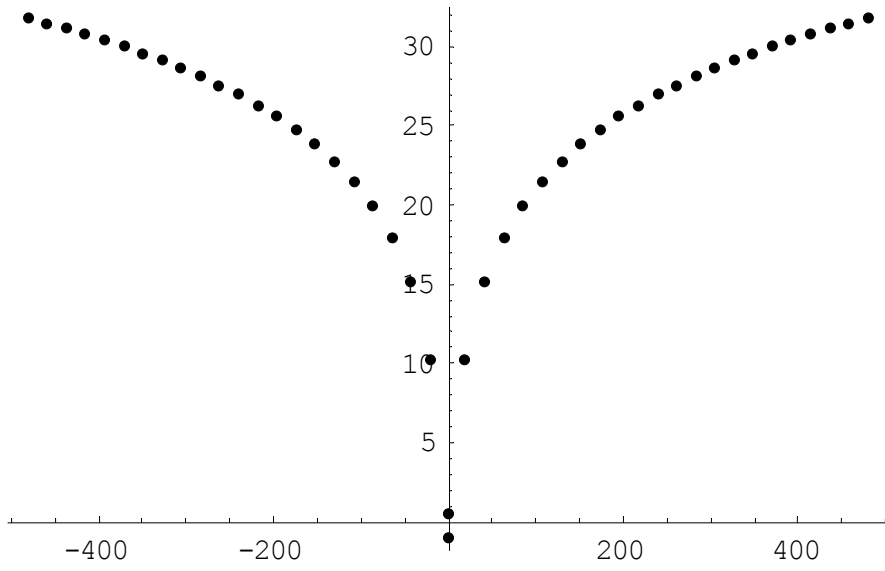
**$\alpha$  is the single lower half-plane root. ( $\text{Im } \alpha < 0$ )**

**$\gamma_i$  are the infinity of upper half-plane roots. ( $\text{Im } \gamma_i > 0$ )**



1.

A plot of  $|f'(z)/f(z)|$  where  $f(z) = r + \psi(z)$  for the Merton jump-diffusion



**Figure 2.**  
The location of some roots of  $r + \psi(z) = 0$  for the point jump model.