“Geometries and Smile Asymptotics for a Class of Stochastic Volatility Models”
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Topics (1-3: in literature; 4: new)

1. What are implied volatility smiles & why asymptotics?

2. The Main Theorem for computation at $T = 0$.

3. General approaches to the computation:
   (i) Compute geodesics.
   (ii) Solve a generalized Eikonal problem.
   (iii) Take a limit with a characteristic function.

4. Elements of the solution for the $CEV(p)$-vol model:
   $(p \in R, \mid \rho \mid < 1)$

\[
\begin{align*}
    dS &= \sqrt{V} S \left\{ \rho dB(1) + \sqrt{1 - \rho^2} dB(2) \right\} \\
    dV &= V^p dB(1)
\end{align*}
\]
Acknowledgments and a few sources

I have greatly benefited from much correspondence with Martin Forde related to today’s topic, whom I thank.
I also thank Greg Egan for ideas on visualization/embedding.

Finance related:
Avellaneda, M. “From SABR to Geodesics”. 2005 slides

Probability related:

Riemannian geometry/physics related:
Option Prices follow the Black-Scholes model with a “custom volatility” – the implied volatility

![Call Option Price vs. Stock Price Graph]

**Black-Scholes formula:**

\[ C_{BS}(S,K,T,\sigma) = S\Phi(d_+) - Ke^{-rT}\Phi(d_-), \]

where \( \Phi(z) = \int_{-\infty}^{z} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \text{cumulative normal} \)

and \[ d_{\pm} = \frac{\log(S / Ke^{-rT}) \pm \sigma^2T / 2}{\sigma\sqrt{T}} \]

**Real-world price:**  \( C_{market} = C_{BS}(S,K,T,\sigma_{implied}) \)

**State-dependent model:**  \( C(S,K,T,\theta) = C_{BS}(S,K,T,\sigma_{implied}) \)

Of course, for this to work: \( \sigma_{implied} = f(S,K,T,\theta) \)
Stochastic Volatility Models

**Working example:** CEV(p)-vol model: \((V = \sigma^2)\)

\[
\begin{align*}
    dS_t &= rS_t dt + \sigma_t S_t dB_t^{(1)} \\
    dV_t &= b(V_t) dt + \xi V_t^p (\rho dB_t^{(1)} + \sqrt{1 - \rho^2} dB_t^{(2)})
\end{align*}
\]

For this class of models:
(Stock price level independent/Translation invariant):

\[
\sigma_{\text{implied}} = f(T, x, y)
\]

where \(x = \log(S/K) = \log(\text{Stock Price}/\text{Strike Price})\)

\(y = V = \text{Stochastic volatility}\)

In general, \(\sigma_{\text{implied}}\) must be numerically computed. But …

The very nice property is that it has a 
formal power series (all diffusions):

\[
(*) \quad \sigma_{\text{implied}} = f^{(0)}(x, y) + T f^{(1)}(x, y) + T^2 f^{(2)}(x, y) + \cdots
\]

**Proof:**
(I) Substitute \(C(S, K, T, \theta) = C_{\text{BS}}(S, K, T, \sigma_{\text{implied}})\)

in the PDE for \(C(S, K, T, \theta)\) (generic n-factor diffusion)

(II) Result is ugly, but ansatz (*) *works* (ugly \(\Rightarrow\) beautiful)
Typical market example (SPX)

$$\sigma_{imp}(T = \frac{1}{12} \text{ yr}, x, y \approx \frac{25\%}{\text{yr}})$$

SPX implied volatility $\sigma_{imp}(x)$, one month-to-go

Today, we explain how to compute the leading $T \to 0$ behavior in stochastic volatility models:

$$\sigma_{imp}(x, y) \triangleq f^{(0)}(x, y) = \lim_{T \to 0} \sigma_{implied}(T, x, y)$$
The Main Theorem for computing Asymptotic Smile

\[ \sigma_{imp}(x, y) = \lim_{T \to 0} \sigma_{implied}(T, x, y) \]

Before stating it, we need a lemma (real proofs: see Varadhan):

Background: the call option price is determined by

\[ C(T, S_0, V_0; K) = e^{-rT} E_{(S_0, V_0)}[(S_T - K)^+] \]

\[ = e^{-rT} \int_0^{S_T} \max[0, S_T - K] q(T, S_0, V_0; S_T) dS_T \]

where the probability transition density

\[ q(T, S_0, V_0; S_T) dS_T = P_{(S_0, V_0)}[S_T \in dS_T]. \]

reflects arriving at the terminal stock price \( S_T \) with any volatility. This is distinguished from the ‘complete’ transition density:

\[ p(T, S_0, V_0; S_T, V_T) dS_T dV_T = P_{(S_0, V_0)}[S_T \in dS_T, V_T \in dV_T] \]

Let’s abbreviate the ‘state variables’ by \( \bar{x}_t = (S_t, V_t) \).

By the Markov property, for any time sub-division \( T = n\Delta t \),

\[ q(T, S_0, V_0; S_T) = \int p(\Delta t, \bar{x}_0; \bar{x}_{t_1}) p(\Delta t, \bar{x}_{t_1}; \bar{x}_{t_2}) \cdots p(\Delta t, \bar{x}_{t_{n-1}}; \bar{x}_{t_n}) \) \( dx_{t_1} \cdots dx_{t_{n-1}} dV_T \)
The Main Theorem (cont)

\[ q(T, S_0, V_0; S_T) = \int p(\Delta t, \vec{x}_0; \vec{x}_{t_1})p(\Delta t, \vec{x}_{t_1}; \vec{x}_{t_2})\cdots p(\Delta t, \vec{x}_{t_{n-1}}; \vec{x}_{t_n}) \, dx_{t_1} \cdots dx_{t_{n-1}} \, dV_T \]

**Heuristic argument:**

\( p(\Delta t, \vec{x}; \vec{y}) \) is the transition density for a \([2D]\) diffusion process with drift \( \vec{b}_t \equiv \vec{b}(x_t) \) and variance-covariance matrix

\[ a_t = [a_{ij}(\vec{x}_t)], \quad (i, j = 1, \ldots, D) \]

For small enough \( \Delta t \), the transition densities must be approximately \( D \)-dimensional Gaussian:

\[ p(\Delta t, \vec{x}; \vec{y}) \approx \frac{1}{(2\pi)^{D/2} (\det a)^{1/2}} \times \exp \left\{ -\frac{1}{2\Delta t} [\vec{y} - \vec{x} - \vec{b}(x) \Delta t]'a^{-1}(x)[\vec{y} - \vec{x} - \vec{b}(x) \Delta t] \right\} \]

To leading order, the drifts \( \vec{b}(x) \Delta t \) don’t contribute:

\[ q(T, S_0, V_0; S_T) \approx \int \exp \left\{ -\frac{1}{2\Delta t} \sum_{i=1}^{n} (x_{t_i} - x_{t_{i-1}})'a^{-1}(x_{t_{i-1}})(x_{t_i} - x_{t_{i-1}}) \right\} \, dx_{t_1} \cdots dx_{t_{n-1}} \, dV_T \]

Note: I am writing \( x_t = D \)-vector with no arrows now \([D = 2]\).
The Main Theorem (cont)

\[ q(T, S_0, V_0; S_T) \approx \]

\[ \int \exp \left\{ -\frac{1}{2\Delta t} \sum_{i=1}^{n} (x_{t_i} - x_{t_{i-1}})' a^{-1}(x_{t_{i-1}})(x_{t_i} - x_{t_{i-1}}) \right\} dx_{t_1} \cdots dx_{t_{n-1}} dV_T \]

In the limit, the points \( \{ x_{t_i} \} \rightarrow \{ x_t \} \) create a continuous path (for any diffusion). The integrand is a maximum along the paths \( \{ x_t \} \) which minimize the sum and becomes concentrated there (saddle point/steepest descent/WKB/etc idea). Interpret

\[ g \equiv a^{-1}(x) = [g_{ij}(x)] \text{ as a metric tensor} \]

With implied sums \((i, j = 1, \cdots , D)\) on upper/lower repeated indices:

\[ \frac{1}{2\Delta t} \sum_{i=1}^{n} (x_{t_i} - x_{t_{i-1}})' a^{-1}(x_{t_{i-1}})(x_{t_i} - x_{t_{i-1}}) \]

\[ = \Delta t \sum_{i=1}^{n} \left[ g(x_{t_{i-1}}) \right]_{jk} \frac{(x_{t_i} - x_{t_{i-1}})^j (x_{t_i} - x_{t_{i-1}})^k}{\Delta t} \]

\[ \rightarrow \left( \Delta t \rightarrow 0 \right) \frac{1}{2T} \int_0^1 g_{ij}(x(s)) \dot{x}^i(s) \dot{x}^j(s) ds, \quad \left[ \dot{x} = \frac{dx}{ds} \right] \]

[using \( \Delta t = (\Delta s)T \), so \( n\Delta t = n\Delta s T = T \rightarrow n\Delta s = 1 \)]
The Main Theorem (cont)

Lemma: \( q(T, S_0, V_0; S_T) \approx \frac{1}{2T} \min_{x(0) = (S_0, V_0), x(1) = (S_T, \text{free})} \int_0^1 g_{ij}(x(s)) \dot{x}^i(s) \dot{x}^j(s) ds \)

\[ 
\uparrow \text{Example of a large deviation principle.}
\]

Example of a geodesic distance function:

Indeed, Varadhan proved, for D-dimensional diffusions, with \( A \) = some set not containing \( x \), that:

\[ P_x [X_T \in A] \approx \exp \left\{ -\frac{d^2(x, A)}{2T} \right\} , \]

where

\[ d^2(x, A) = \min_{\gamma(0) = x, \gamma(1) \in A} \int_0^1 g_{ij}(\gamma(s)) \dot{\gamma}^i(s) \dot{\gamma}^j(s) ds \]

The minimizing paths are geodesics
[in a Riemannian space \((M, g)\)]

Notation: \( \text{Expression}(T) \approx \exp \left\{ -\frac{I(\text{parms})}{T} \right\} \)

means \( I(\text{parms}) = -\lim_{T \to 0} T \log \text{Expression}(T) \).
This accounts for many missing factors!
The Main Theorem (cont)

Recall from an earlier slide:

\[ C(T, S_0, V_0; K) = e^{-rT} \int_0^{S_T} \max[0, S_T - K] q(T, S_0, V_0; S_T) dS_T \]

Since \[ \frac{d^2}{dK^2} \max[0, S_T - K] = \delta(S_T - K) \], (Dirac delta), we have

\[ \frac{d^2}{dK^2} C(T, S_0, V_0; K) = e^{-rT} q(T, S_0, V_0; K) \]

\[ \approx \exp \left\{ - \frac{d^2(x_0, y_0; A_k)}{2T} \right\}, \]

using \[ x_0 = \log S_0, \ y_0 = V_0, \] and \[ d^2(x_0, y_0; A_k) \]

is the geodesic distance to the set \[ A_k := \text{the line } x = k \triangleq \log K \]
in the state space \((x, y)\).

With multi-factor (say \(m\) factors) stochastic volatility models, the state is \((x_t, \tilde{\theta}_t) = (x_t, \theta^1_t, \theta^2_t, \ldots, \theta^m_t)\),

then \(y_0 = \tilde{\theta}_0\), and \(A_k := \text{same } x = k\) (hyperplane now).
The Main Theorem (cont)

\[(1) \quad \frac{d^2}{dK^2} C(T, S_0, V_0; K) \approx \exp \left\{ \frac{-d^2(x_0, y_0; A_k)}{2T} \right\} \]

For the Black-Scholes model, with an out-of-the-money call (S<K),

\[\frac{d^2}{dK^2} C_{BS}(T, S_0, V_0; K) \approx \exp \left\{ \frac{-(x_0 - k)^2}{2V_0 T} \right\} \]

Hence, for a general stochastic volatility model,

\[(2) \quad \frac{d^2}{dK^2} C(T, S_0, V_0; K) \approx \exp \left\{ \frac{-(x_0 - k)^2}{2\sigma_{imp}^2(x_0 - k, y_0)T} \right\} \]

By translation invariance in the \( x \) coordinate:

\[d^2(x_0, y_0; A_k) = d^2(x_0 - k, y_0; A_0),\]
The Main Theorem (cont)

\[
\frac{d^2}{dK^2} C(T, S_0, V_0; K) \approx \exp \left\{ \frac{d^2 (x_0 - k, y_0; A_0)}{2T} \right\} \\
\approx \exp \left\{ -\frac{(x_0 - k)^2}{2\sigma_{imp}^2 (x_0 - k, y_0)T} \right\},
\]

So, comparing, finally yields the main theorem:

Solution to the asymptotic smile problem for diffusions:

\[
\sigma_{imp}^2 (x, y) = \frac{x^2}{d^2 (x, y)}
\]

where \(d(x, y)\) = minimum geodesic distance from \(P = (x, y)\) to the \(y\)-axis. Now \(x\) and \(y\) are scalar coordinates (recall: the financial variables are \(x = \log(S_0 / K)\), and \(y = V_0\)). We have suppressed the dependence on the target set \(A\). The target set is always the \(y\)-axis in the remainder of the presentation.

Pictorial solution:(free endpoint/geodesic) problem:
Hitting the Target: the Local Volatility Connection

Given the metric \( g = [g_{ij}(x, y)] \), and the starting point \( P_0 = (x, y) \), one can compute all the geodesics that pass through \( P_0 \). For reasonably close values of \( x \), one of these geodesics will be the distance minimizer to the target. It hits the target at some optimal \( y = y_1^* \). It can be shown, although we don’t have time today, that

\[
y_1^* = \lim_{T \to 0} E_{(S_0,V_0)}[V_T \mid S_T = K] = \lim_{T \to 0} \alpha(T,S_0,K,V_0) \text{ (the effective local volatility)}
\]

\[
= \lim_{T \to 0} \frac{\int V_T p(T,S_0,V_0;K,V_T)dV_T}{\int p(T,S_0,V_0;K,V_T)dV_T}
\]

Effective local volatility (2D problem is equiv. to 1D): \( C(T,S,K,V) \) solves exactly, for all \( T \),

\[
C_T = \frac{1}{2} \alpha(T,S,K,V)K^2C_{KK} - rKC_K
\]
General approaches to computation

We need the distance function $d(x, y)$. Take the CEV(p)-vol model, for example. The variance-covariance matrix and the metric are

$$a(x, y) = \begin{pmatrix} g^{ij} \end{pmatrix} = \begin{pmatrix} y & \rho y^{p+1/2} \\ \rho y^{p+1/2} & y^{2p} \end{pmatrix}$$

$$g(x, y) = a^{-1} = \begin{pmatrix} g_{ij} \end{pmatrix} = \frac{1}{1 - \rho^2} \begin{pmatrix} y^{-1} & -\rho y^{-p-1/2} \\ -\rho y^{-p-1/2} & y^{-2p} \end{pmatrix}$$

Here are three general methods.

Method (1): Compute all the geodesics $\{ \gamma^i(\tau) \} = \{ X(\tau), Y(\tau) \}$ leaving $(x, y)$. The geodesic equations are well-known:

$$\frac{d^2 \gamma^i}{d\tau^2} + \Gamma^i_{jk}(\gamma) \frac{d \gamma^j}{d\tau} \frac{d \gamma^k}{d\tau} = 0, \quad (i = 1, 2)$$

where the Christoffel symbol components are given by

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{im}(x) \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right)$$
General approaches to computation (Geodesic method)

There are two constants of the motion:

(i) **kinematic condition:** \( \frac{(\dot{X})^2}{Y} + \frac{(\dot{Y})^2}{Y^{2p}} = 1 \)

(ii) **conserved x-momentum:** \( \dot{X} = \pm \sqrt{k} Y \) \((k = \text{const})\)

Note on the latter: Whenever the metric is independent of a coordinate \( x^i \), there is a conserved momentum: \( U_i = g_{ij} U^j = g_{ij} \gamma^j \). In our case, the metric has no \( x \)-dependence and so \( U_1 = g_{11} \gamma^1 = \dot{X}(\tau)/Y(\tau) = \text{const} \), (taking transformed orthogonal coordinates or \( \rho = 0 \)).

Thus, there is a one parameter family of geodesics from \((x, y)\) to the target. One of these \((k = k^*)\) is the distance minimizer. The main complication is that the vertical distance to “infinity” is bounded for \( p > 1 \). Moreover, this vertical move can sometimes be the shortest way to the target. It is straightforward to show:

**Theorem (Point-to-target distance bound):**
Consider the standardized base point \( P_0 = (x,1) \).
Then, under the CEV(p)-vol metric,
\[
d(x,1) \leq \frac{1}{p-1} < \infty, \quad (p > 1 \text{ and } |\rho| < 1)
\]
This bound reflects moving along a vertical geodesic to \( \infty \)
General approaches to computation (Geodesic method)

**CEV(p)-vol model solution** \((\rho = 0, \ p \in R)\)

First, define the function \(F_p(k) \triangleq B_{1-k}(1/2, 1-p)\),
where \(B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt\) (Incomplete Beta).
I use \(G_p(k) = k^{p-3/2}F_{p-1}(k)\), and \(H_p(k) = k^{p-1}F_p(k)\).

**Basic Solution System** \((-\infty < p < 3/2)\)

**Step I:** set \(z = |x|y^{p-3/2}\) and
solves for the root \(k = k(z)\) that solves:
\[z = G_p(k)\].

**Step II:** Then \(d(x,y) = y^{1-p}H_p(k)\)

**Modified Solution System** \((3/2 \leq p < \infty)\)

First, calculate the critical values pair:

\[k_{\text{crit}} = \max_{0 \leq k \leq 1} \left\{ k : H_p(k) = 1/(p-1) \right\}\]
\[z_{\text{crit}} = G_p(k_{\text{crit}})\]

Then, if \(z < z_{\text{crit}}\), use the Basic Solution System, otherwise:

\[d(x,y) = y^{1-p} \times \begin{cases} H_p(k), & (z < z_{\text{crit}}) \\ 1/(p-1), & (z \geq z_{\text{crit}}) \end{cases}\]
General approaches to computation (Eikonal eqn)

Method (2): Solve a generalized Eikonal problem. Abbreviating \( \partial_i d \equiv \partial d / \partial x^i \), where \( x^1 = x \) and \( x^2 = y \), the Eikonal/Hamilton-Jacobi eqn is:

\[
a^{ij} (\partial_i d)(\partial_j d) = 1 \text{ with bound cond: } d(x = 0, y) = 0
\]

i.e.

\[
y d_x^2 + 2\rho y^{p+1/2} d_x d_y + y^{2p} d_y^2 = 1
\]

This is the fastest way to a number. The trick is to note that there is a scaling form solution:

\[
d(x, y) = y^{1-p} F(z), \text{ where } z = x y^{p-3/2}
\]

This yields, using \( \alpha = p - 3/2 \), the non-linear ODE:

\[
[1 + 2\rho \alpha z + \alpha^2 z^2 ](F')^2 + 2(1 - p)[\rho + \alpha z] FF' + (1 - p)^2 F^2 = 1
\]

Easily solved numerically in Mathematica -- use the StoppingTest option to handle the critical \( z \)-values, where \( F = 1/(\rho - 1) \). This ODE also forms the starting point for a quasi-analytic solution for general \( (\rho, p) \) that extends the \( \rho = 0 \) solution given in these slides. For details, see:

General approaches to the computation (Char. Func.)

Method (3): Take a limit with a characteristic function. If you already know the characteristic function

\[ \Phi(T, z, V_0) = E_{(S_0, V_0)}[e^{iz \log(S_T / S_0)}], \]

you can rescale it and find \( d(x, y) \) from a Legendre transform/saddle point.

This does not help us directly with the general CEV(p)-vol model, as the characteristic functions are known only for half-integers:

\[ p = \frac{1}{2}, 1, \frac{3}{2} \text{ (Heston, GARCH/SABR, } \frac{3}{2} \text{ model)} \]

But, multi-factor Heston-type models are common in finance, and this may be the most direct route to the asymptotics for those applications.

(Details: see Martin Forde’s recent paper: arXiv:math.PR/0609117)